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Dilation Theorems for Positive Definite Operator Kernels Having Majorants

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Dilation theorems for Banach space valued stochastic processes and operator valued positive definite kernels are considered. It is shown, e.g., that a Banach space valued stochastic process X can be dilated to another process Y, if and only if the covariance kernel of Y is a majorant of the covariance kernel of X. Positive definite operator kernels having majorants of certain special type are characterized.

INTRODUCTION

We are concerned with the dilation theory for Banach space valued stochastic processes and positive definite operator kernels with values in the space $\bar{L}(E, E')$ consisting of all bounded antilinear operators from a complex Banach space E into its dual E'.

Let T be a semigroup. It has been recently shown that a positive definite operator kernel $B: T \times T \to \bar{L}(E, E')$ satisfies the boundedness condition

$$\sum_{j=1}^{n} \sum_{k=1}^{n} (B(st_j, st_k)f_j)(f_k) \leqslant \rho(s) \sum_{j=1}^{n} \sum_{k=1}^{n} (B(t_j, t_k)f_j)(f_k), \qquad (*)$$

$s, t_j \in T, f_j \in E, j = 1,..., n$; $n \in N$,for some function $\rho: T \to R_+$, if and only if it is the covariance kernel of a Banach space valued stochastic process X on

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T with the property that the propagator of X exists, i.e., there exists a representation π_X of T in the Hilbert space $\overline{\mathrm{sp}}\{X\}$ such that $X_{st} = \pi_X(s)X_t$ (cf. [6, 10]).

In this paper we consider the possibility of dilating a given Banach space valued stochastic process X to another process Y with nicer properties than X, e.g., having a propagator. We show that a Banach space valued stochastic process X can be dilated to another process Y, if and only if the covariance kernel K_Y of Y is a majorant of the covariance kernel K_X of X, i.e.,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} (K_X(t_j, t_k)f_j)(f_k) \leqslant \sum_{j=1}^{n} \sum_{k=1}^{n} (K_Y(t_j, t_k)f_j)(f_k), \qquad (**)$$

$t_j \in T$, $f_j \in E$, $j = 1,..., n$; $n \in N$. Especially, positive definite kernels $Q: G \times G \to C$, defined on a locally compact Abelian group G, for which there exists a continuous positive definite function $Q': G \to C$ majorizing Q are characterized.

The boundedness condition (*) is closely related to the general Sz.-Nagy's [18] dilation theorem (see also [9]). It also arises in the dilation theory of Banach space valued stochastic processes (cf. [6, 7, 10, 15, 16, 20]). The inequality (**) arises in the dilation theory for certain non-stationary stochastic processes (cf. [2, 8, 12]) and vector measures (cf. [8, 11]). One of the main motivations for this study has been to analyze the relationship between the conditions (*) and (**). One might expect that the latter dilation theory, related to orthogonally scattered dilations of bounded vector measures, could be deduced from Sz.-Nagy's dilation theorem. We show that, in general, such implication does not hold (cf. Example 13 in Section 3).

1. Positive Definite Operator Kernels Having Majorants

Let E be a complex Banach space. By $\bar{L}(E, E')$ we denote the space of all bounded antilinear operators from E in the (topological) dual E' of E. Furthermore, $L(E, F)$ stands for the space of all bounded linear operators from E into another Banach space F. We write $L(F) = L(F, F)$.

Let Z be a (fixed) set. Recall that a mapping $B: Z \times Z \to \bar{L}(E, E')$ is positive definite, if

$$\sum_{j=1}^{n} \sum_{k=1}^{n} (B(t_j, t_k)f_j)(f_k) \geqslant 0 \qquad (1)$$

for all $t_j \in Z, f_j \in E, j = 1,..., n$; $n \in N$.

The following proposition was presented in [19] (cf. [4, 6, 7, 14]).

PROPOSITION 1 (Aronszajn–Kolmogorov). *A mapping $B: Z \times Z \to \bar{L}(E, E')$ is positive definite, if and only if there exist a Hilbert space H and a mapping $X: Z \to L(E, H)$ such that*

$$B(s, t) = X_t^* X_s \qquad \textit{for all} \quad s, t \in Z. \tag{2}$$

Remark. As usual, we have identified the dual H' of a Hilbert space H with H by an antilinear isometrical isomorphism. This identification is always included in a representation of the form (2).

Remark. Suppose $B: Z \times Z \to \bar{L}(E, E')$ is positive definite and suppose $X: Z \to L(E, H)$ is a mapping satisfying (2). If

$$H = \bigvee_{s \in Z} X_s(E), \tag{3}$$

then H is called *minimal.* If $\tilde{X}: Z \to L(E, \tilde{H})$ is another mapping satisfying (2) and if $\tilde{H}$ is also minimal, then $X, \tilde{X}$ and H, $\tilde{H}$, respectively, are unitarily equivalent (cf. [19]).

EXAMPLE 2. Suppose $Q: Z \times Z \to C$ is a positive definite complex valued kernel, i.e.,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k Q(t_j, t_k) \geqslant 0$$

for all $t_j \in Z$, $a_j \in C$, $j = 1,..., n$; $n \in N$. For $t \in Z$ put

$$Q_t(s) = Q(s, t), \qquad s, t \in Z. \tag{4}$$

Let $H(Q)$ be the reproducing kernel Hilbert space associated with Q (cf. [1]). Recall that $Q_t \in H(Q)$, $t \in Z$; and $(Q_t, Q_s)_{H(Q)} = Q(s, t)$, $s, t \in Z$.

Identify $H(Q)$ with $L(C, H(Q))$ by identifying $y \in H(Q)$ with the mapping $a \to ay$, $a \in C$; and identify C with $\bar{L}(C, C)$ by identifying $a \in C$ with the mapping

$$(a(f))(g) = a\bar{f}g, \qquad f, g \in C.$$

Then, $Q: Z \times Z \to C$ $(=\bar{L}(C, C))$ is a positive definite operator kernel. Furthermore, one representation for Q in the form (2) is $Q(s, t) = X_t^* X_s$, $s, t \in Z$, if $X: Z \to H(Q)$ $(=L(C, H(Q)))$ is defined by $X_t = Q_t$, $t \in Z$. In this case $H(Q)$ is minimal.

DEFINITION 3. A positive definite operator kernel $B': Z \times Z \to \bar{L}(E, E')$ is called a *majorant* of a given positive definite kernel $B: Z \times Z \to \bar{L}(E, E')$, if $B' - B$ is positive definite. In this case we write $B' \succ B$.

Let H be a Hilbert space and let $X: Z \to L(E, H)$ be a mapping. In what follows we often write

$$K_X(s, t) = X_t^* X_s, \qquad s, t \in Z;$$

$$\overline{\text{sp}}\{X\} = \bigvee_{s \in Z} X_s(E).$$

The following dilation theorem is closely related to the dilation theorems for bounded vector measures with values in a Hilbert space [11; Theorem 13] (cf. [8; Corollary 6]) and certain non-stationary stochastic processes [12; Theorem 11] (cf. [8; Theorem 5]), respectively.

THEOREM 4. *Let H_1, H_2 be Hilbert spaces and let $X: Z \to L(E, H_1)$, $Y: Z \to L(E, H_2)$ be mappings. Then $K_X \prec K_Y$, if and only if there exist a Hilbert space H and isometries $J_1: \overline{\text{sp}}\{X\} \to H$, $J_2: \overline{\text{sp}}\{Y\} \to H$ such that*

$$J_1 X_s = P_{J_1(\overline{\text{sp}}\{X\})} J_2 Y_s, \qquad s \in Z, \tag{5}$$

where $P_{J_1(\overline{\text{sp}}\{X\})}$ is the orthogonal projection of H onto the closed linear subspace $J_1(\overline{\text{sp}}\{X\})$ in H.

Proof. Suppose (5) holds. Then for any $n \in N$, $t_j \in E, j = 1,..., n$, one has

$$\begin{aligned} \sum_{j=1}^{n} \sum_{k=1}^{n} (K_X(t_j, t_k) f_j)(f_k) &= \left\| \sum_{j=1}^{n} X_{t_j}(f_j) \right\|^2 \\ &\leqslant \left\| \sum_{j=1}^{n} Y_{t_j}(f_j) \right\|^2 \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} (K_Y(t_j, t_k) f_j)(f_k), \end{aligned}$$

proving that $K_X \prec K_Y$.

On the other hand, suppose $K_X \prec K_Y$. Put $U = E \times Z$ and define functions $Q: U \times U \to C$, $Q': U \times U \to C$ by

$$Q(u, v) = (K_X(s, t) f)(g), \qquad Q'(u, v) = (K_Y(s, t) f)(g),$$

where $u = (f, s)$, $v = (g, t)$; $f, g \in E$; $s, t \in Z$. Since K_X and K_Y are positive definite operator kernels, Q and Q' are positive definite kernels on $U \times U$. Since $K_X \prec K_Y$, the function $Q'': U \times U \to C$ defined by

$$Q''(u, v) = Q'(u, v) - Q(u, v), \qquad u, v \in U,$$

is also a positive definite kernel on $U \times U$.

Consider the reproducing kernel Hilbert spaces $H(Q)$ and $H(Q'')$ associated with Q and Q'', respectively (cf. Example 2). For $s \in Z$ define, by applying the notation introduced in Example 2,

$$X_s'(f) = Q_{(f,s)}, \qquad f \in E.$$

Then $X_s' \colon E \to H(Q)$ is a well-defined linear mapping. Moreover,

$$\|X_s'(f)\|^2 = (K_X(s,s)f)(f) \leqslant \|K_X(s,s)\| \, \|f\|^2, \qquad f \in E,$$

which shows that $X_s' \colon E \to H(Q)$ is bounded. In a similar way one can show that the mapping $V_s' \colon E \to H(Q'')$ defined by $V_s'(f) = Q''_{(f,s)}$, $f \in E$, is a bounded linear operator.

Put $H = H(Q) \oplus H(Q'')$. For $s \in Z$ define a bounded linear operator $Y_s' \colon E \to H$ by

$$Y_s'(f) = (X_s'(f); V_s'(f)), \qquad f \in E.$$

Then,

$$K_X = K_{X'}, \qquad K_Y = K_{Y'}.$$

Thus, there exist isometries $J_1 \colon \overline{\mathrm{sp}}\{X\} \to H$, $J_2 \colon \overline{\mathrm{sp}}\{Y\} \to H$ such that

$$J_1 X_s(f) = (X_s'(f); 0); \qquad J_2 Y_s(f) = Y_s'(f), \qquad f \in E, \quad s \in Z.$$

Clearly, $J_1(\overline{\mathrm{sp}}\{X\}) = \overline{\mathrm{sp}}\{X'\} = H(Q) \oplus \{0\}$. Thus,

$$\begin{aligned} J_1 X_s(f) &= (X_s'(f); 0) \\ &= P_{J_1(\overline{\mathrm{sp}}\{X\})}(X_s'(f); V_s'(f)) \\ &= P_{J_1(\overline{\mathrm{sp}}\{X\})} J_2 Y_s(f), \qquad f \in E, \quad s \in Z. \end{aligned}$$

The theorem is proved.

DEFINITION 5. Let H_1, H_2 be Hilbert spaces. If the mappings $X \colon Z \to L(E, H_1)$, $Y \colon Z \to L(E, H_2)$ satisfy (5), then Y is called a *dilation* of X.

The following corollaries are immediate consequences of Theorem 4.

COROLLARY 6. *Let H_1, H_2 be Hilbert spaces. A mapping $Y \colon Z \to L(E, H_2)$ is a dilation of a given mapping $X \colon Z \to L(E, H_1)$, if and only if $K_X \prec K_Y$.*

COROLLARY 7. *Let* $B \colon Z \times Z \to \bar{L}(E, E')$, $B' \colon Z \times Z \to \bar{L}(E, E')$, *be* *positive definite operator kernels. Then $B \prec B'$, if and only if there exist a Hilbert space H and a mapping $X\colon Z \to L(E, H)$ such that*

(i) $B'(s, t) = K_X(s, t) = X_t^* X_s$, $s, t \in Z$;

(ii) $B(s, t) = X_t^* P_M X_s$, $s, t \in Z$;

where P_M is the orthogonal projection of H onto a closed linear subspace M in H.

2. Majorants Satisfying a Boundedness Condition

Let H be a Hilbert space and let T be a semigroup. A family $\pi_X\colon T \to L(\overline{\mathrm{sp}}\{X\})$ is called a *propagator* of a given mapping $X\colon T \to L(E, H)$, if

$$\pi_X(s) X_t(f) = X_{st}(f), \qquad f \in E; \quad s, t \in T$$

(cf. [6, 7, 10]). The propagator $\pi_X\colon T \to L(\overline{\mathrm{sp}}\{X\})$ is uniquely determined, provided that it exists. It is obvious that the propagator $\pi_X\colon T \to L(\overline{\mathrm{sp}}\{X\})$ is a representation of T, i.e., $\pi_X(s)\pi_X(t) = \pi_X(st)$, $s, t \in T$. If T is unital with a unit e, then π_X is unital, i.e., $\pi_X(e) = I$.

The following proposition summarizes results by several authors (cf. [6, 7, 10, 20] and references therein).

Proposition 8. *Let H be a Hilbert space, let T be a semigroup and let $X\colon T \to L(E, H)$ be given. Then:*

(i) *K_X satisfies the boundedness condition*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} (K_X(st_j, st_k) f_j)(f_k) \leqslant \rho(s) \sum_{j=1}^{n} \sum_{k=1}^{n} (K_X(t_j, t_k) f_j)(f_k) \qquad \text{(B.C)}$$

$s, t_j \in T, f_j \in E, j = 1,..., n$; $n \in N$, for some function $\rho\colon T \to R_+$, if and only if the propagator $\pi_X\colon T \to L(\overline{\mathrm{sp}}\{X\})$ of X exists;

(ii) *if T is in addition unital, then K_X satisfies* (B.C) *if and only if there exists a unital representation $\pi\colon T \to L(\overline{\mathrm{sp}}\{X\})$ and $R \in L(E, \overline{\mathrm{sp}}\{X\})$ such that*

$$K_X(s, t) = R^* \pi(t)^* \pi(s) R, \qquad s, t \in T. \tag{6}$$

One can choose $R = X_e$, $\pi = \pi_X$.

Remark. (i) Suppose T is a $*$-semigroup. We say that a mapping $B\colon T \to \bar{L}(E, E')$ is *positive definite*, if the mapping $B'\colon T \times T \to \bar{L}(E, E')$,

$B'(s,t) = B(t^*s)$, $s, t \in T$, is positive definite, i.e., B' satisfies (1). Equivalent forms of the corresponding boundedness condition for positive definite $B: T \to \bar{L}(E, E')$ have been presented in [15, 16].

(ii) The propagator $\pi_X: T \to L(\overline{\mathrm{sp}}\{X\})$ of a given $X: T \to L(E, H)$ is a $*$-representation, if and only if K_X satisfies (B.C) and there exists a positive definite $B: T \to \bar{L}(E, E')$ such that

$$K_X(s,t) = B(t^*s), \qquad s, t \in T,$$

or equivalently, if and only if there exists a representation $\pi: T \to L(\overline{\mathrm{sp}}\{X\})$ satisfying (6) which is also a $*$-representation (cf. [20; Theorem 2.4, Corollary 2.6]).

(iii) Suppose, in addition, T is a group (and $s^* = s^{-1}$, $s \in T$). Then (B.C) is trivially satisfied for any $B': T \times T \to \bar{L}(E, E')$ of the form $B'(s,t) = B(t^*s)$, $s, t \in T$, if $B: T \to \bar{L}(E, E')$ is positive definite; one can choose $\rho \equiv 1$. In this case any $*$-representation $\pi: T \to L(\overline{\mathrm{sp}}\{X\})$ satisfying (6) is, in fact, a unitary reresentation of T.

THEOREM 9. *Let T be a unital semigroup (resp. a $*$-semigroup). A positive definite operator kernel $B: T \times T \to \bar{L}(E, E')$ has a majorant $B': T \times T \to \bar{L}(E, E')$ satisfying* (B.C) *(and $B'(s,t) = B''(t^*s)$, $s, t \in T$, for some $B'': T \to \bar{L}(E, E')$), if and only if there exist a Hilbert space H, $R \in L(E, H)$ and a unital representation (resp. $*$-representation) $\pi: T \to L(H)$ such that*

$$B(s,t) = R^*\pi(t)^* P_M \pi(s) R, \qquad s, t \in T, \tag{7}$$

where P_M is an orthogonal projection of H onto a closed linear subspace M in H.

Proof. We present a proof just in the case T is a unital semigroup, since the case T is a $*$-semigroup can be handled in a similar way.

Suppose (7) holds. Put

$$B'(s,t) = R^*\pi(t)^*\pi(s)R, \qquad s, t \in T.$$

It then follows from Proposition 8 that B' satisfies (B.C). Furthermore, if the mapping $X: T \to L(E, H)$ is defined by $X_s = \pi(s)R$, $s \in T$. Then $B' = K_X$ and, a fortiori, it follows from Corollary 7 that $B \prec B'$.

On the other hand, suppose a positive definite operator kernel $B': T \times T \to \bar{L}(E, E')$ satisfies (B.C) and $B \prec B'$. Let $X: T \to L(E, H_1)$ and $Y: T \to L(E, H_2)$ be such that

$$K_X = B \qquad \text{and} \qquad K_Y = B',$$

respectively (cf. Proposition 1). Moreover, let

$$B'(s, t) = K_Y(s, t) = R^*\pi(t)^*\pi(s)R, \qquad s, t \in T,$$

be a representation in the form (6). Since one can choose $Y_s = \pi_s R$ (cf. Proposition 8(ii)) and since

$$K_X = B \prec B' = K_Y,$$

the latter part of the theorem follows immediately from Theorem 4.

The next result follows immediately from Theorem 9, Corollary 6 and Theorem 4.

COROLLARY 10. *Let T be a unital semigroup (resp. a *-semigroup) and let H be a Hilbert space. A mapping $X: T \to L(E, H)$ has a dilation $Y: T \to L(E, \tilde{H})$ having a propagator π_Y (which is a *-representation), if and only if K_X has a majorant $B: T \times T \to \bar{L}(E, E')$ satisfying* (B.C) (*and, in addition, if $B(s, t) = B'(t^*s)$, $s, t \in T$, for some $B': T \to \bar{L}(E, E')$*).

Remark. By applying the methods used in [4] one may observe that analogous results hold for locally convex vector spaces E with the so-called factorization property and only for such spaces.

3. APPLICATIONS

We consider the case when $E = C$ and T is a locally compact Abelian group G with a (fixed) Haar measure λ and the dual group Γ.

Our first applications are based on the following characterization, which is due to Kluvánek [5; Theorem 2]: A continuous and bounded function $x: G \to F$ with values in a semi-reflexive Banach space F is the Fourier–Stieltjes transform of a (regular) bounded F-valued vector measure on Γ, if and only if there exists an $M \geqslant 0$ such that

$$\left\| \int_G x(t)\, u(t)\, d\lambda(t) \right\| \leqslant M \sup |\hat{u}| \qquad \text{for all} \quad u \in L^1(G); \tag{8}$$

here $\hat{u}$ stands for the Fourier transform of $u \in L^1(G)$.

We make use also of a dilation theorem for bounded (regular) vector measures on a locally compact Hausdorff space S [11; Theorem 13] (cf. [8; Corollary 6]). It states that any bounded (regular) vector measure μ on S with values in a Hilbert space H has an orthogonally scattered dilation (cf.

(5)), or equivalently, there exists a bounded positive (regular) measure ν on S such that

$$\left\| \int_S u \, d\mu \right\|^2 \leqslant \int_S |u|^2 \, d\nu \tag{9}$$

for all bounded Borel functions $u: S \to C$.

The following theorem characterizes positive definite kernels $Q: G \times G \to C$ majorized by a continuous positive definite function $Q': G \to C$.

THEOREM 11. *For a positive definite kernel $Q: G \times G \to C$ there exists a continuous positive definite function $Q': G \to C$ satisfying*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k Q(t_j, t_k) \leqslant \sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k Q'(t_k^{-1} t_j) \tag{M.C$'$}$$

for all $a_j \in C$, $t_j \in G$, $j = 1, \ldots, n$; $n \in N$, if and only if Q is continuous and bounded and there exists a constant $M \geqslant 0$ such that

$$\int_G \int_G Q(s, t) \overline{u(t)} \, d\lambda(s) \, d\lambda(t) \leqslant M (\sup |\hat{u}|)^2 \tag{10}$$

for all $u \in L^1(G)$.

Proof. Suppose (M.C$'$) holds. Let $H(Q)$ and $H(Q')$ be the reproducing kernel Hilbert spaces associated with Q and Q', respectively (cf. Example 2). Since (M.C$'$) holds, it follows from Theorem 4 that the mappings $Q_s \in H(Q)$, $s \in G$, and $Q'_s \in H(Q')$, $s \in G$ (cf. Example 2), satisfy (5). Thus, the continuity and boundedness of Q' imply the continuity and boundedness of Q. Furthermore, by (5)

$$\left\| \int_G Q_s u(s) \, d\lambda(s) \right\| \leqslant \left\| \int_G Q'_s u(s) \, d\lambda(s) \right\|$$

for all $u \in L^1(G)$. Since $Q': G \to C$ is positive definite and continuous, it follows from Bochner's theorem that Q' is the Fourier–Stieltjes transform of a bounded positive (regular) measure ν on Γ. Thus, by applying Fubini's theorem we get

$$\begin{aligned} \left\| \int_G Q'_s u(s) \, d\lambda(s) \right\|^2 &= \int_G \int_G Q'(t^{-1} s) u(s) \overline{u(t)} \, d\lambda(s) \, d\lambda(t) \\ &= \int_\Gamma |\hat{u}(\gamma)|^2 \, d\nu(\gamma) \leqslant \nu(G) (\sup |\hat{u}|)^2 \end{aligned}$$

for all $u \in L^1(G)$, proving the first part of the theorem.

On the other hand, suppose a continuous and bounded positive definite kernel $Q: G \times G \to C$ satisfies (10). Since (10) holds, it is obvious that the continuous and bounded mapping $Q_s \in H(Q)$, $s \in G$ (cf. Example 2), satisfies (8) and, a fortiori, there exists a bounded (regular) vector measure μ on Γ with values in $H(Q)$ such that

$$Q_s = \int_\Gamma \overline{\gamma(s)}\, d\mu(\gamma), \qquad s \in G$$

[5; Theorem 2]. Let ν be a bounded positive (regular) measure on Γ satisfying (9) for μ (cf. [11; Theorem 13]). Define a continuous positive definite function $Q': G \to C$ by

$$Q'(s) = \int_\Gamma \overline{\gamma(s)}\, d\nu(\gamma), \qquad s \in G.$$

Since (9) holds, it is obvious that Q and Q'satisfy (M.C′).

The theorem is proved.

Remark. Theorem 11 can be interpreted also as follows: A second order stochastic process $x: G \to H$ has a continuous stationary dilation, if and only if x is continuous, bounded and satisfies (8), or equivalently, if and only if x is the Fourier transform of a bounded (regular) vector measure (cf. [8, Theorem 5; 12, Theorem 11]).

A stochastic process version of the following theorem was presented in [13; Theorem 4] in the case $G = R$.

THEOREM 12. *Suppose a continuous and bounded positive definite kernel $Q: G \times G \to C$ satisfies the boundedness condition*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k Q(st_j, st_k) \leqslant \rho(s) \sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k Q(t_j, t_k), \tag{B.C′}$$

s, $t_j \in G$, $a_j \in C$, $j = 1,\ldots, n$, $n \in N$, *for a constant function $\rho \equiv M \geqslant 0$. Then there exists a continuous positive definite function $Q': G \to C$ satisfying* (M.C′).

Proof. Consider the reproducing kernel Hilbert space $H(Q)$ and the mapping $Q_s \in H(Q)$, $s \in G$, associated with Q (cf. Example 2). Since (B.C′) is satisfied the mapping Q_s, $s \in G$, has a propagator $\pi_Q: G \to L(H(Q))$ (cf. Proposition 8). Since $\rho \equiv M$, it follows that $\|\pi_Q(s)\| \leqslant M$, $s \in G$. Furthermore, since π_Q is a commuting group of operators, it follows from a general form of a result by Sz.-Nagy [17] (cf. [3; p. 35]) that there exist $B \in L(H(Q))$

with a bounded inverse and a unitary representation $\pi' \colon G \to L(H(Q))$ such that $\pi_Q = B^{-1}\pi' B$.

Since π' is unitary and since Q is bounded and continuous, it follows that the function $P \colon G \to C$ defined by

$$P(s) = (BQ_s, BQ_e)_{H(Q)}, \qquad s \in G,$$

is continuous and positive definite. As in the first part of the proof of Theorem 11 we then get

$$\begin{aligned}\int_G \int_G Q(s,t) u(s) \overline{u(t)} \, d\lambda(s) \, d\lambda(t) &= \left\| \int_G Q_s u(s) \, d\lambda(s) \right\|^2 \\ &\leqslant \|B\|^2 \left\| \int_G P_s u(s) \, d\lambda(s) \right\|^2 \\ &\leqslant \|B\|^2 P(e) (\sup |\hat{u}|)^2\end{aligned}$$

for all $u \in L^1(G)$. Thus, all hypothesis of Theorem 11 are satisfied and, a fortiori, the theorem follows from Theorem 11.

We close this section by showing that (M.C′) (resp. (B.C′)) can be satisfied even if (B.C′) (resp. (M.C′)) is not satisfied.

EXAMPLE 13. (a) Define a continuous positive definite kernel $Q \colon R \times R \to C$ by

$$Q(s,t) = e^s e^t, \qquad s, t \in R.$$

It is obvious that (B.C′) is satisfied. However, (M.C′) cannot be satisfied, since Q is not bounded.

(b) It follows from Theorem 11 that any (non-trivial) positive definite kernel $Q \colon R \times R \to C$ of the form

$$Q(s,t) = r(s) \overline{r(t)}, \qquad s, t \in R,$$

where r is the Fourier–Stieltjes transform of a bounded (regular) complex valued measure on R satisfies (M.C′). However, such a Q does not satisfy (B.C′), if, e.g., $r(0) = 0$.

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